Thus, covariance and form-invariance are distinct concepts. Transformations ensuring covariance of field equations in general include transformations between different admissible systems of reference not equally suited for the description of physical phenomena. In contrast to this, transformations ensuring form-invariance of the metric tensor of space—time (and hence also form-invariance of equations) include transformations only between systems of reference which are equivalent from a physical point of view: in these systems of reference all physical phenomena occur in the same way with corresponding initial and boundary conditions.

Since the geometry of space—time in passing between different systems of reference does not change and remains pseudo-Euclidean, for any system of reference, inertial or noninertial, there exists a 10-parameter group of coordinate transformations leaving the metric form-invariant. Thus, in pseudo-Euclidean space—time for any system of reference we can find an infinite collection of other systems of reference, transformations between which leave the metric form-invariant.

This means that in pseudo-Euclidean space—time a generalized principle of relativity formulated in [8-9] holds: for any physical system of reference we choose (inertial or noninertial) it is always possible to find an infinite collection of other systems of reference so that all physical phenomena in them occur in the same way as in the initial system of reference, so that we do not and cannot possess the means to distinguish by experimental in which reference system of this infinite collection we are located.

Thus, Minkowski's geometry has general character, being the natural geometry for all known fields and thus guaranteeing for them that the generalized principle of relativity is satisfied. Pseudo-Euclidean space—time is not a priori given from the start with an independent existence. Its existence is inseparable from the existence of matter, since this is the geometry in which the evolution of matter takes place.

## 10. Connection of Conservation Laws with the Geometry of Space—Time

The possibility of obtaining conservation laws for a closed system of interacting fields depends to large extent on the character of the geometry of space—time.

As is known [1, 2], the construction of a theory of any physical field can be carried out on the basis of a Lagrangian formalism. In this case the physical field is described by some function of coordinates and time called the field function, the equations for which can be obtained from the variational principle of stationary action. Aside from the field equations, the Lagrangian path to constructing a classical theory of wave fields provides the possibility of obtaining a number of differential relations called differential conservation laws. These relations are consequences of the invariance of the action function under coordinate transformations of space—time and relate the local dynamic characteristics of the field and their covariant derivatives in the geometry natural for them.

At present, it is customary in the literature to distinguish two types of differential conservation laws: strong and weak. A strong law is usually a differential relation which is satisfied because of invariance of the action function under coordinate transformations and does not require that the equations of motion of the field be satisfied. Weak conservation laws can be obtained from strong conservation laws by considering the equations of motion for the system of interacting fields. An example of a weak conservation law is the covariant equation (2.17) of conservation of the energy—momentum tensor of matter in Riemannian space—time. This equation was obtained as a consequence of the requirement of invariance of the action function of matter (2.10) under any infinitely small coordinate transformation (2.12) and the condition that the equations of motion of matter be satisfied (2.11).

It should be emphasized that, in spite of the name, differential conservation laws in general do not assert conservation of anything either locally or globally. They are simply differential identities connecting various characteristics of the field which hold because the action function does not change under an arbitrary coordinate transformation (i.e., it is a scalar). These relations received their name from the analogy with the corresponding differential conservation laws in pseudo-Euclidean space—time where the corresponding integral laws can be obtained from the differential conservation laws. Thus, for example, writing the law of conservation of the total energy—momentum tensor of interacting fields [1, 2] in the Cartesian coordinate system of pseudo-Euclidean space—time, we have

$$\frac{\partial}{\partial t} t^{0i} + \frac{\partial}{\partial x^\alpha} t^{\alpha i} = 0.$$

Integrating this equality over some volume and using the Gauss–Ostrogradskii theorem, we obtain

$$\frac{d}{dt}\int dVt^{0i}=-\oint t^{\alpha i}dS_{\alpha}.$$

This relation implies that the change of the energy–momentum of a system of interacting fields in some volume is equal to the flux of energy–momentum through the surface bounding this volume. If there is no flux of energy–momentum through the surface

$$\oint t^{\alpha i}dS_{\alpha}=0,$$

then we arrive at the law of conservation of the total four-momentum of an isolated system

$$\frac{d}{dt}P^{i}=0,$$

where

$$P^{i}=\frac{1}{c}\int t^{0i}dV.$$

Analogous integral relations in pseudo-Euclidean space–time can also be obtained for the angular momentum.

In an arbitrary Riemannian space–time the presence of a differential covariant conservation equation does not guarantee the possibility of obtaining a corresponding integral conservation law.

The possibility of obtaining integral conservation laws in an arbitrary Riemannian space–time is entirely predetermined by its geometry and is closely connected with the existence of Killing vectors of the given space–time or, as is sometimes said, with the presence of a group of motions in Riemannian space–time. We shall consider this question in somewhat more detail, since the formalism developed here can be used to obtain integral conservation laws in arbitrary curvilinear coordinate systems of pseudo-Euclidean space–time.

In an arbitrary Riemannian space–time by a method similar to that expounded in Sec. 2 it is possible to obtain a covariant equation of conservation of the total energy–momentum tensor of the system:

$$\nabla_{l}T^{ml}=\partial_{l}T^{ml}+\Gamma^{l}_{ln}T^{nm}+\Gamma^{m}_{nl}T^{nl}=0. \qquad (10.1)$$

We multiply this equation by a Killing vector, i.e., by a vector $\eta_m$ satisfying the Killing equations

$$\nabla_{n}\eta_{m}+\nabla_{m}\eta_{n}=0. \qquad (10.2)$$

Because of the symmetry of the tensor $T^{nm}$, the expression obtained can be written in the form

$$\eta_{m}\nabla_{l}T^{ml}=\nabla_{l}[\eta_{m}T^{ml}]=0.$$

Using the properties of the covariant derivative, from this we have

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{l}}[\sqrt{-g}T^{ml}\eta_{m}]=0.$$

Since the left side of this equality is a scalar, we can multiply it by $\sqrt{-g}dV$ and integrate over some volume. As a result, we obtain the integral conservation law in Riemannian space–time

$$\frac{d}{dx^{0}}\int\sqrt{-g}T^{0m}\eta_{m}dV=-\oint dS_{\alpha}\sqrt{-g}T^{\alpha m}\eta_{m}. \qquad (10.3)$$

If there is no flux of the three-dimensional vector $T^{\alpha m}\eta_m$ through the surface bounding the volume, then

$$\int\sqrt{-g}T^{0m}\eta_{m}dV=\text{const}. \qquad (10.4)$$

Thus, if Killing vectors exist from the differential conservation law (10.1) it is possible to obtain integral conservation laws (10.3), (10.4).

We shall now determine under what conditions on the metric of Riemannian space–time the Killing equations (10.2) have solutions, i.e., under what conditions there exists a vector satisfying Eqs. (10.2). We first observe that the Killing equations (10.2) are a consequence of the condition that the Lie variation of the metric tensor of Riemannian space–time under infinitesimal coordinate transformations (2.12) vanish.

Indeed, comparing expressions (2.16) and (10.2), we see that the Killing equations require that the Lie variation of the metric tensor of Riemannian space–time vanish:

$$\delta_L g_{ni}=0.$$

Thus, the Killing vectors describe infinitely small coordinate transformations leaving the metric form-invariant.

The Killing equations (10.2) represent a system of linear partial differential equations of first order. According to the general theory [17], to determine conditions for the integrability of a system of partial differential equations it is necessary to reduce it to the form

$$\frac{\partial \Theta^a}{\partial x^i}=\psi_i{}^a(\Theta^b,\ x^n), \tag{10.5}$$

where $\Theta^a$ are unknown functions; i, n = 1, 2,...,N; $a$ = 1, 2,...,M. Then the integrability condition for system (10.5) can be obtained from the equality

$$\frac{\partial^2\Theta^a}{\partial x^i\partial x^n}=\frac{\partial^2\Theta^a}{\partial x^n\partial x^i},$$

by replacing the partial derivatives of first order by the right side of Eqs. (10.5):

$$\frac{\partial\psi_i{}^a}{\partial x^n}+\frac{\partial\psi_i{}^a}{\partial\Theta^b}\psi_n{}^b=\frac{\partial\psi_n{}^a}{\partial x^i}+\frac{\partial\psi_n{}^a}{\partial\Theta^b}\psi_i{}^b. \tag{10.6}$$

If the integrability conditions (10.6) are satisfied identically by virtue of Eqs. (10.5), then system (10.5) is called completely integrable, and its solution contains M parameters — the maximum possible number of arbitrary constants for the given system.

If system (10.5) is not completely integrable, then its solution will contain a fewer number of arbitrary constants. We shall determine under what conditions the solution of the Killing equations (10.2) in a Riemannian space $V_n$ contains the maximum possible number of parameters and what this number is.

We shall carry out all computations in an explicitly covariant form which is a covariant generalization of the scheme presented above for finding integrability conditions for a system of partial differential equations. For this we first reduce the Killing equations (10.2) to the required form. We covariantly differentiate the Killing equations (10.2) with respect to the variable $x^n$. As a result, we obtain

$$\eta_{i;jn}+\eta_{j;in}=0.$$

Because of this equation we have

$$\eta_{i;jn}+\eta_{j;in}+\eta_{i;nj}+\eta_{n;ij}-\eta_{j;ni}-\eta_{n;ji}=0.$$

Regrouping terms in this expression, we obtain

$$\eta_{i;jn}+\eta_{i;nj}+(\eta_{j;in}-\eta_{j;ni})+(\eta_{n;ij}-\eta_{n;ji})=0. \tag{10.7}$$

On the other hand, by the commutation rule for covariant derivatives, we have

$$\eta_{i;nj}-\eta_{i;jn}=\eta_h R^h_{inj}. \tag{10.8}$$

Substituting expression (10.8) into relation (10.7), we obtain

$$2\eta_{i;jn}+\eta_h R^h_{inj}+\eta_h R^h_{jin}+\eta_h R^h_{nij}=0. \tag{10.9}$$

Using the Ricci identity

$$R^h_{inl}+R^h_{nli}+R^h_{lin}=0, \tag{10.10}$$

we have

$$\eta_h R^h_{inj}+R^h_{jin}\eta_h=\eta_h R^h_{nij}.$$

Therefore, expression (10.9) can be written in the form

$$\eta_{i;jn}=-\eta_h R^h_{nij}.$$

We thus have the following variant equations:

$$\eta_{i;n}+\eta_{n;i}=0,\quad \eta_{i;jn}=-R^h_{nij}\eta_h. \tag{10.11}$$

We shall transform this system of covariant differential equations into a system containing only first covariant derivatives. For this together with the n unknown components of the vector $\eta_m$ we introduce the unknown tensor $\lambda_{mi}$ according to the equations

$$\eta_{i;m}=\lambda_{im}. \tag{10.12}$$

This tensor contains $n^2$ unknown components but of these only $n(n-1)/2$ components are independent, since this tensor is antisymmetric by Eqs. (10.2) and (10.12):

$$\lambda_{mi}+\lambda_{im}=0. \tag{10.13}$$

Considering all this, the desired system of covariant differential equations has the form

$$\eta_{m;i}=\lambda_{mi},\quad \lambda_{mi;j}=\eta_h R^h_{jim}. \tag{10.14}$$

Thus, we have brought the Killing equations (10.2) to a system of special form consisting of linear differential equations solved for the covariant derivatives of first order.

This system is a covariant generalization of system (10.5) whereby the role of the unknown functions $\Theta^a$ is played by the $n(n+1)/2$ components of the tensors $\lambda_{mi}$ and $\eta_m$:

$$\Theta^a=\{\eta_m,\ \lambda_{mi}\}.$$

The integrability condition for system (10.14) can be obtained from the commutation rule of covariant derivatives which is a consequence of the independence of the order of the derivatives in partial differentiation. On the basis of this rule, we have

$$\begin{gathered}\eta_{i;mj}-\eta_{i;jm}=\eta_h R^h_{imj},\\ \lambda_{im;jl}-\lambda_{im;lj}=\lambda_{ih}R^h_{mjl}+\lambda_{hm}R^h_{ijl}.\end{gathered} \tag{10.15}$$

Replacing the first covariant derivatives on the left sides of these equalities by their expressions (10.14) and using the property (10.13) that the tensor $\lambda_{im}$ is antisymmetric, we obtain the inequality conditions for system (10.14) in the form

$$\lambda_{im;j}-\lambda_{ij;m}=\eta_h R^h_{imj}, \tag{10.16}$$

$$[\eta_h R^h_{jmi}]_{;l}-[\eta_h R^h_{lmi}]_{;j}=\lambda_{ih}R^h_{mjl}+\lambda_{hm}R^h_{ijl}. \tag{10.17}$$

It is easy to see that the first of these expressions is satisfied identically by Eqs. (10.14) of the system and the properties of the curvature tensor. Thus, if condition (10.17) is identically satisfied by virtue of only properties of the symmetry of Riemannian spacetime, then system (10.14) will be completely integrable, and hence the solution of the Killing equations (10.2) will contain the maximum possible number $M = n(n+1)/2$ of arbitrary constants. Since the unknown functions $\eta_i$ and $\lambda_{mi} = -\lambda_{in}$ contained in system (10.14) must hereby be independent, the left side of expression (10.17) vanishes provided that the following conditions are satisfied:

$$R^h_{mlj;l}-R^h_{lij;m}=0, \tag{10.18}$$

$$\delta_j{}^s R^h_{iml}-\delta_j{}^h R^s_{iml}-\delta_i{}^s R^h_{jml}+\delta_i{}^h R^s_{jml}+\delta_l{}^s R^h_{mij}-\delta_l{}^h R^s_{mij}-\delta_m{}^s R^h_{lij}+\delta_m{}^h R^s_{lij}=0. \tag{10.19}$$

Contracting expressions (10.19) on the indices $l$ and s with consideration of the relations

$$R^s_{ims}=R_{im};\quad R^s_{smi}=0$$

and the Ricci identity (10.10) gives

$$(n-1)R^h_{mij}=\delta_j{}^h R_{mi}-\delta_i{}^h R_{jm}.$$

From this it follows that

$$R_{lmij}=\frac{1}{n-1}(g_{jl}R_{mi}-g_{il}R_{jm}). \tag{10.20}$$

Multiplying this equality by $g^{mi}$, we obtain

$$nR_{jl}=g_{jl}R.$$

Substituting this relation into expression (10.20), we obtain a condition that equality (10.19) be satisfied identically:

$$R_{lmij}=\frac{R}{n(n-1)}[g_{jl}g_{mi}-g_{il}g_{jm}]. \tag{10.21}$$

From expression (10.21) and Eq. (10.18) we obtain the condition that the scalar curvature should satisfy:

$$[\delta_j^h g_{im}-\delta_i^h g_{jm}]\frac{\partial}{\partial x^l}R-[\delta_j^h g_{il}-\delta_i^h g_{lj}]\frac{\partial}{\partial x^m}R=0.$$

Multiplying this relation by $\delta_h^l g^{mi}$, we have

$$(n-1)\frac{\partial R}{\partial x^j}=0.$$

Since in the case we consider $n > 1$, in order that this condition be satisfied it is necessary and sufficient that R = const. Hence, the integrability conditions (10.18) and (10.19) for the Killing equations (10.2) will be identically satisfied if and only if the curvature tensor of Riemannian space time has the form

$$R_{lmij}=\frac{R}{n(n-1)}[g_{jl}g_{mi}-g_{il}g_{jm}],$$

where R = const.

Hence, the Killing equations have solutions containing the maximum possible number $M = n(n + 1)/2$ of arbitrary constants (parameters) if and only if the Riemannian space $V_n$ is a space of constant curvature. If the space $V_n$ is not a space of constant curvature, then the number of parameters will be less.

Hence, from a mathematical point of view the presence of integral conservation laws of energy—momentum and angular momentum is a reflection of particular properties of space—time: its homogeneous and isotropic properties. There exist three types of four-dimensional spaces [15] possessing the properties of homogeneity and isotropicity to the extent that they admit the introduction of 10 integrals of the motion for a closed system: a space of constant negative curvature (Lobachevskii space), a space of zero curvature (pseudo-Euclidean space), and a space of constant positive curvature (the space of Riemann). The first two spaces are infinite, having infinite volume, while the third space is finite, having finite volume but no boundaries.

We shall now find a Killing vector in an arbitrary curvilinear coordinate system of pseudo-Euclidean space—time. For this we first write the Killing equations in a Cartesian coordinate system:

$$\frac{\partial}{\partial x^i}\eta_m+\frac{\partial}{\partial x^m}\eta_i=0.$$

Hence, for determining Kiling vectors we have a system of 10 linear partial differential equaequations of first order.

Solving this system by the usual rules, we obtain

$$\eta_i=a_i+\omega_{im}x^m, \tag{10.22}$$

where $a_i$ is an arbitrary constant, infinitely small vector and $\omega_{mi}$ is an arbitrary constant, infinitely small tensor satisfying the condition

$$\omega_{im}=-\omega_{mi}.$$

Thus, the solution (10.22), as was to be expected, contains all 10 arbitrary parameters.

Since expression (10.22) contains 10 independent parameters, we actually have 10 independent Killing vectors, and relation (10.22) is a linear combination of these 10 independent vectors.

We shall clarify the meaning of these parameters. Substituting the expression (10.22) into relation (2.12), we obtain

$$x'^i = a^i + \omega_m{}^i x^m + x^i. \tag{10.23}$$

It is evident from this expression that the four parameters $a^i$ are the components of a four-vector of infinitely small translations of the system of reference. The three parameters $\omega_{\alpha\beta}$ are components of a tensor of rotation by an infinitely small angle about some axis (so-called pure rotations). The three parameters $\omega_{0\beta}$ describe infinitely small rotations in the plane $x^0x^\beta$ — so-called Lorentz rotations. Since the metric tensor $\gamma_{mi}$ is form-invariant under translations, pseudo-Euclidean space—time is homogeneous; its properties do not depend on at what point of space—time the origin of the coordinates is placed. Similarly, form-invariance of the metric tensor $\gamma_{mi}$ under four-dimensional rotations implies its isotropicity. This means that in pseudo-Euclidean space—time all directions are equivalent.

Thus, pseudo-Euclidean space—time admits a 10 parameter group of motions consiting of a four-parameter subgroup of translations and a six-parameter subgroup of rotations. The presence of this group of motion and the existence of the corresponding Killing vectors guarantees the presence of 10 integral conservation laws of energy—momentum and angular momentum of a system of interacting fields.

Indeed, noting that in a Cartesian coordinate system $\sqrt{-g} = 1$, from the general relation (10.3) in the case of the subgroup of translations ($\eta_i = a_i$) we have

$$\frac{d}{dx^0}\int T^{0m} a_m dV = -\oint dS_\alpha T^{\alpha m} a_m.$$

Since $a_m$ is an arbitrary constant vector, from this equality we have

$$\frac{d}{dx^0}\int T^{0m} dV = -\oint dS_\alpha T^{\alpha m}.$$

For an isolated system of interacting fields the expression on the right side of this relation is equal to zero as a result of which its total four-momentum is conserved:

$$P^m = \int T^{0m} dV = \text{const}. \tag{10.24}$$

Altogether analogously, for

$$\eta_i = \omega_{im} x^m$$

we obtain

$$\frac{d}{dx^0}\int dV T^{0m} x^i \omega_{mi} = -\oint dS_\alpha T^{\alpha m} x^i \omega_{mi}.$$

Since the constant tensor $\omega_{mi}$ is antisymmetric, from this we obtain the integral conservation law of angular momentum:

$$\frac{d}{dx^0}\int dV\,[T^{0m}x^i - T^{0i}x^m] = -\oint dS_\alpha\,[T^{\alpha m}x^i - T^{\alpha i}x^m]. \tag{10.25}$$

For an isolated system its total angular momentum is conserved due to the vanishing of the right side of equality (10.25):

$$M^{mi} = \int dV\,[T^{0m}x^i - T^{0i}x^m] = \text{const}. \tag{10.26}$$

It should be noted that we can obtain the solution of Killing's equations (10.2) in arbitrary curvilinear coordinates of pseudo-Euclidean space—time from the solution (10.23) of these equations in a Cartesian coordinate system because of the tensorial character of the quantities $x^i$ and $\eta^i$. For this we go over in expression (10.23) from Cartesian coordinates $x^i$ to arbitrary curvilinear coordinates $x^i_H$:

$$x^i = f^i(x_H).$$

We then obtain

$$\eta_m^H = \frac{\partial f^i}{\partial x_H^m}\eta_i(x(x_H)).$$

Thus, in an arbitrary curvilinear coordinate system of pseudo-Euclidean space—time the Killing vector has the form

$$\eta_m = \frac{\partial f^i(x_H)}{\partial x_H^m}a_i + \frac{\partial f^i(x_H)}{\partial x_H^m}\omega_{is}f^s(x_H). \tag{10.27}$$

Generalization of expressions (10.24)-(10.26) to the case of arbitrary curvilinear coordinates presents no difficulty. Proceeding exactly as above, for the four-momentum of an isolated system we obtain

$$P^i = \int \sqrt{-\gamma(x_H)}\,dx_H^1 dx_H^2 dx_H^3 T^{0m}(x_H)\frac{\partial f^i(x_H)}{\partial x_H^m}.$$

The antisymmetry tensor of angular momentum in this case has the form

$$M^{im} = \int \sqrt{-\gamma(x_H)}\,dx_H^1 dx_H^2 dx_H^3 T^{0s}(x_H)\left[f^m(x_H)\frac{\partial f^i(x_H)}{\partial x_H^s} - f^i(x_H)\frac{\partial f^m(x_H)}{\partial x_H^s}\right].$$

Thus, the possibility of obtaining integral conservation laws depends on the character of the geometry of space—time. In the case of four dimensions (physical space—time) only spaces of constant curvature possess all 10 integral conservation laws, while in other spaces the number of them is less than 10.

## 11. A Field Approach to the Description of Gravitational Interaction

In order that the gravitational field may be considered a physical field in the spirit of Faraday—Maxwell with its usual properties of a carrier of energy—momentum, it suffices for us to select as a natural geometry for the gravitational field the geometry of a space of constant curvature. Since experimental data obtained in studying the strong, weak, and electromagnetic interactions bear witness to the fact that for fields connected with these interactions the natural geometry of space—time is pseudo-Euclidean, at least at the present stage of our knowledge it may be assumed that this geometry is the sole natural geometry for all physical processes including gravitational processes.

This assertion constitutes one of the basic propositions of the field approach to the theory of gravitational interaction we developed. It is altogether obvious that it will lead to fulfillment of all conservation laws of energy—momentum and angular momentum, ensuring the existence of all 10 integrals of the motion for a system consisting of the gravitational field and the remaining fields of matter. The gravitational field in the field approach, similar to all other physical fields, is characterized by its energy—momentum tensor which contributes to the total energy—momentum tensor of the system. This is the basic difference of our approach from Einstein's theory. It should be noted that, in addition to general simplicity, in pseudo-Euclidean space—time the integration of tensor quantities has an altogether definite meaning.

Another key question arising in the construction of a theory of the gravitational field is the question of the nature of the interaction of the gravitational field with matter. In acting on matter the gravitational field can effectively alter its geometry if it enters the terms for the highest derivatives in the equations of motion of matter. The motion of material bodies and other physical fields in pseudo-Euclidean space—time under the action of the gravitational field will then be indistinguishable from their motion in some effective Riemannian space—time. Universality of the action of the gravitational field on matter follows from experimental data, and hence the effective Riemannian space—time will be the same for all matter.

This leads us to an assertion which we call the identity principle (the principle of geometrization) which is defined as follows: The equations of motion of matter under the action of a gravitational field in pseudo-Euclidean space—time with metric tensor $\gamma_{ni}$ can be